Appendix for Hooper chapter in Temporal Patterns in the Nervous System, E. Covey, ed.

To show that, when temporal summation has stabilized, V_{ave} = *I* x *R* x *duty cycle*, first remember that the average of a function *f(x)* is given by $ave_{f(x)} = \frac{1}{b-a}\int_a^b f(x)dx$. When temporal summation has stabilized, the function to be averaged has two parts, the charging curve from V_{min} to V_{max} that occurs during the current pulse, and the relaxation curve from V_{max} to V_{min} that occurs during the interpulse interval (Fig. 1). Since current pulse duration (*PD*) plus interpulse interval (*IPI*) equals cycle period (*per*), it follows that the average is given by $ave = \frac{1}{per}\left(\int_0^{PD} rise(t)dt + \int_0^{IPI} relax(t)dt\right)$. (1)

The functions *rise(t)* and *relax(t)* can be determined as follows. For *rise(t)*, first note that if there were no charge on the capacitor when the current was turned on, *V* would increase according to $IR(1-e^{-t/RC})$. However, in the case at hand when the current pulse begins, there is already a charge (V_{min}) on the capacitor. It is therefore necessary to re-integrate the differential equation that describes an RC circuit not from *V* to 0, but instead from *V* to V_{min}.

$$I_{inj} = I_R + I_C = V/R + CdV/dt$$

Rearranging gives

$$\frac{dV}{IR-V} = \frac{dt}{RC}$$

We now integrate the *V* side of the equation from *V* to V_{min}, and the *time* side of the equation from *t* to 0 (because, once the temporal stabilization is over, the charging curve exactly repeats each cycle). This gives

$$\int_{V_{\min}}^{V} \frac{dV}{IR - V} = \int_{0}^{t} \frac{dt}{RC}$$

$$-\left[\ln(IR - V\right]_{V_{\min}}^{V} = \left[\frac{t}{RC}\right]_{0}^{t}$$

$$\ln(IR - V) - \ln(IR - V_{\min}) = -t / RC$$

$$\ln\left(\frac{IR - V}{IR - V_{\min}}\right) = -t / RC$$

$$\frac{IR - V}{IR - V_{\min}} = e^{-t / RC}$$

$$V = IR(1 - e^{-t / RC}) + V_{\min} e^{-t / RC} \qquad (2)$$

This is how V changes during the current pulse, and is therefore the *rise(t)* function needed in equation 1. The *relax(t)* function is simple exponential relaxation from V_{max},

$$V = V_{\max}(e^{-t / RC}) . \qquad (3)$$

Substitution into equation 1 gives

$$ave = \frac{1}{per}\left(\int_{0}^{PD} IR(1 - e^{-t / RC})dt + \int_{0}^{PD} V_{\min} e^{-t / RC} dt + \int_{0}^{IPI} V_{\max} e^{-t / RC} dt\right).$$

Integrating gives

$$ave = \frac{1}{per}\left(IR \cdot PD + IR \cdot RC(e^{-PD / RC} - 1) - V_{\min} RC(e^{-PD / RC} - 1) - V_{\max} RC(e^{-IPI / RC} - 1)\right)$$

.

Since *PD/per* is duty cycle, it follows that if

$$IR \cdot RC(e^{-PD / RC} - 1) - V_{\min} RC(e^{-PD / RC} - 1) - V_{\max} RC(e^{-IPI / RC} - 1) = 0$$

then

$$ave = IR \cdot duty\ cycle$$

which is our goal. Factoring out the common RC term gives

$$IR \cdot (e^{-PD/RC} - 1) - V_{\min}(e^{-PD/RC} - 1) - V_{\max}(e^{-IPI/RC} - 1) = 0 \tag{4}$$

as what needs to be shown. To do this requires expressing V_{min} and V_{max} in terms of the circuit's and pattern's fundamental constituents, *I*, *R*, *C*, *PD*, *per*, and *IPI*. This can be done by noting that, when temporal summation is finished, the change in amplitude as the circuit rises from V_{min} to V_{max} equals the change in amplitude as the circuit relaxes from V_{max} to V_{min}. For the rise, V_{max} is reached at time PD, and thus by equation 2,

$$V_{\max} = IR(1 - e^{-PD/RC}) + V_{\min} e^{-PD/RC}.$$

The rise amplitude (? V_{rise}) is $V_{max} - V_{min}$, or

$$\Delta V_{rise} = IR(1 - e^{-PD/RC}) + V_{\min} e^{-PD/RC} - V_{\min} = (IR - V_{\min})(1 - e^{-PD/RC}).$$

Turning to the relaxation amplitude change, the relaxation occurs for the duration of the interpulse interval (*IPI*). Substituting into equation 3 gives $V_{\min} = V_{\max} e^{-IPI/RC}$. Therefore,

$$\Delta V_{relax} = V_{\max} - V_{\max} e^{-IPI/RC} = V_{\max}(1 - e^{-IPI/RC}).$$

Setting $\Delta V_{rise} = \Delta V_{relax}$ gives

$$(IR - V_{\min})(1 - e^{-PD/RC}) = V_{\max}(1 - e^{-IPI/RC})$$

Noting again that $V_{\min} = V_{\max} e^{-IPI/RC}$ and substituting gives,

$$(IR - V_{\max} e^{-IPI/RC})(1 - e^{-PD/RC}) = V_{\max}(1 - e^{-IPI/RC}).$$

Solving for V_{max},

$$V_{\max} = \frac{IR(1 - e^{-PD/RC})}{1 - e^{-IBI/RC} e^{-PD/RC}}$$

Remembering the law of exponents ($e^a e^b = e^{a+b}$) and noting that *IBI* + *PD* = *per* gives

$$V_{\max} = \frac{IR(1-e^{-PD/RC})}{1-e^{-per/RC}}. \tag{5}$$

Substituting this expression into $V_{\min} = V_{\max} e^{-IPI/RC}$ and again using the law of exponents and *IBI* + *PD* = *per* gives

$$V_{\min} = \frac{IR(e^{-IBI/RC} - e^{-per/RC})}{1-e^{-per/RC}}. \tag{6}$$

Substituting equations 5 and 6 into equation 4 gives

$$IR \cdot e^{-PD/RC} - IR - \frac{IR(e^{-IPI/RC} - e^{-per/RC})}{1-e^{-per/RC}}(e^{-PD/RC} - 1) - \frac{IR(1-e^{-PD/RC})}{1-e^{-per/RC}}(e^{-IPI/RC} - 1) = 0$$

. Factoring out the common *IR* term, multiplying both sides of the equation with $1-e^{-per/RC}$, and multiplying through gives

$$(e^{-PD/RC} - e^{(-PD-per)/RC} - 1 + e^{-per/RC}) - (e^{(-IPI-PD)/RC} - e^{-IPI/RC} - e^{(-per-PD)/RC} + e^{-per/RC})$$
$$- (e^{-IPI/RC} - 1 - e^{(-PD-IPI)/RC} + e^{-PD/RC}) = 0$$

Gathering like terms gives

$$e^{-PD/RC} - e^{-PD/RC} - e^{(-PD-per)/RC} + e^{(-per-PD)/RC} - 1 + 1 + e^{-per/RC} - e^{-per/RC}$$
$$- e^{(-IPI-PD)/RC} + e^{(-PD-IPI)/RC} + e^{-IPI/RC} - e^{-IPI/RC} = 0$$

and thus $ave = IR \cdot duty\ cycle$.

Figure 1